# Numerical stability criteria for differential-algebraic systems

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**Abstract**

In this paper we transfer classical results concerning Lyapunov stability of stationary solutions $x_*$ to the classes of DAEs being most interesting for circuit simulation, thereby keeping smoothness as low as possible. We formulate all criteria in terms of the original equation. Those simple matrix criteria for checking regularity, Lyapunov stability etc. are easily realized numerically.

**Key words:** Differential algebraic systems, Lyapunov stability

## 1 Introduction

This paper deals with autonomous quasilinear differential-algebraic equations (DAEs)

(1.1) $$A(x)x' + g(x) = 0,$$

where the leading coefficient matrix $A(x)$ is everywhere singular but has constant rank.
From a geometric point of view, (1.1) should induce a smooth vectorfield on a certain state manifold. However, if it does so, the vectorfield as well as the manifold are given implicitly only, and they are not available in practice for higher index DAEs except for interesting case studies. This is why we insist on terms of (1.1) for numerical stability criteria.
In this paper we transfer classical results concerning Lyapunov stability of stationary solutions of regular ordinary differential equations (ODEs) to the case of DAEs (1.1).
Due to analytic techniques we keep the smoothness as low as possible while the concept of understanding DAEs as differential equations on manifolds supposes more smoothness than it seems to be natural.
For instance, the semi-explicit DAE

(1.2) $$u' - \varphi(u, v) = 0,$$

(1.3) $$\Psi(u, v) = 0,$$

with $C^1$ functions $\varphi, \Psi$ has index 1 if $\Psi'_v(u, v)$ remains nonsingular. Clearly, due to the Implicit Function Theorem, exactly one solution of (1.2), (1.3) passes through each consistent initial point $u_0, v_0$, that is, $\Psi(u_0, v_0) = O$. Locally, (1.2), (1.3) is equivalent with

(1.4) $$u' = \varphi(u, f(u)),$$

(1.5) $$v = f(u),$$

whereby $f(u)$ and $\varphi(u, f(u))$ depend continuously differentiably on $u$.
On the other hand, the geometric concept understands (1.2), (1.3) to be the linearly implicitly given vectorfield

(1.6) $$u' - \varphi(u, v) = 0,$$

$$(1.7) \qquad \Psi_u'(u,v)u' + \Psi_v'(u,v)v' = 0,$$

on the manifold

$$\mathcal{M} := \{(u^\top, v^\top)^\top \in I\!R^m : \Psi(u,v) = 0\}.$$

To arrive with (1.6)), (1.7) at a $C^1$ vectorfield again, we should assume $\Psi \in C^2$.
Moreover, the explicit regular ODE resulting from (1.6), (1.7), namely

$$(1.8) \qquad u' = \varphi(u,v),$$

$$(1.9) \qquad v' = -\Psi_v'(u,v)^{-1}\Psi_u'(u,v)\varphi(u,v),$$

is called the underlying ODE of the DAE (1.2), (1.3). Considering this ODE on the whole space $I\!R^m$ instead on the manifold $\mathcal{M}$ would not be helpful for answering stability questions since the asymptotics of (1.8), (1.9) on the whole of $I\!R^m$ might show a different behaviour than its restriction to $\mathcal{M}$. So a stationary solution being stable on $\mathcal{M}$ may become unstable on $I\!R^m$.

Sometimes it is easier to deal with DAEs having even a constant leading coefficient matrix, say

$$(1.10) \qquad \tilde{A}\tilde{x}' + \tilde{g}(\tilde{x}) = 0.$$

Hence, instead of considering the original DAE (1.1) we may turn to the enlarged systems

$$(1.11) \qquad Px' - y = 0,$$

$$(1.12) \qquad A(x)y + g(x) = 0,$$

or

$$(1.13) \qquad x' - y = 0,$$

$$(1.14) \qquad A(x)y + g(x) = 0,$$

which have obviously the form (1.10). Using system (1.11),(1.12) we always assume $A(x)$ to have a constant nullspace $N = \ker A(x)$, and $P$ stands for any projection matrix with $\ker P = N$. If $\ker A(x)$ depends on $x$ (that is it rotates with varying $x$) we use (1.13), (1.14).
It is well known that enlarging (1.1) to (1.11), (1.12) leaves the index invariant. However, in contrary, the index of system (1.13), (1.14) becomes higher than that of the original DAE (1.1)
At this place it should be mentioned that we are basing on the tractability index (e.g. [1], [2]) defined in terms of the Jacobians of the functions $A, g$ and $\tilde{A}, \tilde{g}$, respectively. Recall that the tractability index represents a generalization of the Kronecker index. Moreover, for the DAEs being discussed in the following, the tractability index is shown to coincide with the differentiation index as well as with the geometric one, supposed the latter exists. This is why we use simply the notion of an *index*.
In circuit simulation the charge oriented modelling leads to DAEs of the form

$$(1.15) \qquad A\frac{d}{dt}C(x) + g(x) = 0,$$

or, equivalently, to the system

$$(1.16) \qquad Ay' + g(x) = 0,$$

$$(1.17) \qquad y - C(x) = 0.$$

Thereby, $A$ is a constant matrix. The system (1.16), (1.17) is somewhat easier to integrate than

$$(1.18) \qquad AC'(x)x' + g(x) = 0$$

since one can perform Newton iterations without second derivatives of $C$.
Obviously, system (1.16), (1.17) has a constant leading matrix, i.e. it is of the form (1.10).

Moreover, the enlarged DAE (1.16), (1.17) has index 1 iff (1.18) has so or is a regular ODE, supposed the condition

$$im\, A \equiv im\, AC'(x) \tag{1.19}$$

is satisfied. Often $C'(x)$ is nonsingular, hence (1.19) is given trivially.
In the following we provide stability criteria for DAEs of the form (1.10) in terms of $\tilde{A}$ and $\tilde{g}'(\tilde{x}_*)$, where $\tilde{x}_*$ denotes the stationary solution under discussion. Clearly, all those results can be traced back for (1.1) resp. (1.15) immediately.

# 2 Basic linear algebra

Given two matrices $A, B \in L(I\!R^m)$ we form the matrix chain

$$A_0 := A, \quad B_0 := B,$$

$$A_{j+1} := A_j + B_j Q_j, \quad B_{j+1} := B_j P_j,\ j \geq 1. \tag{2.1}$$

Thereby, $Q_j \in L(I\!R^m)$ stands for any projector onto $\ker A_j$, and $P_j := I - Q_j$.

**Lemma 2.1** *The matrix pencil $\lambda A + B$ is regular with index $\mu$ if and only if $A_\mu$ is nonsingular but $A_j, j = 0, \ldots, \mu - 1$ are not*

The proof is referred to in [3].

**Lemma 2.2** *For given regular index $\mu$ pencil $\lambda A + B$ the projections $Q_0, \ldots, Q_{\mu-1}$ can be chosen such that*

$$P_0 \cdots P_{\mu-1} = \hat{A}^D \hat{A} \tag{2.2}$$

*becomes true, where $\hat{A}^D$ denotes the Drazin inverse of $\hat{A} := (cA + B)^{-1} A$.*

The proof is given in [4].
Recall (e.g. [5]) that $\hat{A}^D \hat{A}$ respresents the spectral projection onto the finite eigenspace of the pencil along the infinite one.
As a consequence (cf. [5]) of Lemma 2.2, the linear constant coefficient DAE

$$Ax' + Bx = 0 \tag{2.3}$$

is equivalent with

$$(P_0 \cdots P_{\mu-1} x)' + M P_0 \cdots P_{\mu-1} = 0, \tag{2.4}$$

$$x = P_0 \cdots P_{\mu-1} x, \tag{2.5}$$

where

$$M := P_0 \cdots P_{\mu-1} A_\mu^{-1} B. \tag{2.6}$$

Any finite eigenvalue of the pencil is an eigenvalue of $M$ simultaneously. The corresponding eigenspaces belong to $\operatorname{im} P_0 \cdots P_{\mu-1}$. Moreover, $M$ has the zero eigenvalue corresponding to $\ker P_0 \cdots$
$P_{\mu-1} \subset \ker M$.
So, asking for the stability of (2.3) we check the eigenvalues of $M$ or, equivalently, those of the pencil.
The matrix chain (2.1) can be computed numerically without special difficulties. It may be considered as a practical tool for index checking and regularity tests e.g. during the numerical integration. In particular, transforming

$$HA\Pi = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}, \quad R_{11} \text{ nonsingular}, \tag{2.7}$$

say, by a Housholder matrix $H$ and a permutation matrix $\Pi$, we know

$$Q = \Pi \begin{pmatrix} 0 & -R_{11}^{-1}R_{12} \\ 0 & I \end{pmatrix} \Pi^{-1}$$

to be a projection onto $\ker A$.
Thus, the main work we have to do is performing (2.7).
There is some good experience with realizing those tests ([8]). Note that in the framework of index reduction techniques (e.g. [5], [7], [3]) one has to make similar efforts at each step.

# 3 Lyapunov stability

First of all let us recall the famous Theorem of Lyapunov that we will generalize for DAEs.
Consider the regular ODE

$$x' + g(x) = 0, \tag{3.1}$$

which is assumed to have the stationary solution $x_*$, i.e.
$g(x_*) = 0$.

**Theorem 3.1** *(e.g. [9]): Let $g \in C^2(\mathcal{D}, \mathbb{R}^m), \mathcal{D} \subset \mathbb{R}^m$ open, $x_* \in \mathcal{D}, g(x_*) = 0$. If all eigenvalues of the matrix $-B, B := g'(x_*)$, have negative real parts, the equilibrium point $x_*$ is asymptotically stable.*

In particular, stability in the sense of Lyapunov inlcudes the solvability of all initial value problems with initial values $x_0 \in B(x_*, \tau), \tau > 0$ sufficiently small, and all those solutions have continuations up to infinity.
How to apply this result to the nonlinear index 1 and index 2 DAEs

$$Ax' + g(x) = 0 \tag{3.2}$$

we are interested in?
Again, $x_*$ is a stationary solution iff

$$g(x_*) = 0. \tag{3.3}$$

However, now we have to compare them with consistent intial values only, that is, with $x_0 \in B(x_*, \tau) \cap \mathcal{M}$, where $\mathcal{M}$ denotes the corresponding state manifold.
If (3.2) has index 1 on $\mathcal{D}, \mathcal{D} \subset \mathbb{R}^m$ open, it simply holds

$$\mathcal{M} = \{x \in \mathcal{D} : g(x) \in im\, A\}.$$

In that case, the nullspace $\ker A$ and the tangent space

$$S_0(x_*) := T_{x_*}\mathcal{M} = \{z \in \mathbb{R}^m : g'(x_*)z \in im\, A\}$$

intersect trivially, thus

$$\ker A \oplus S_0(x_*) = \mathbb{R}^m.$$

This makes clear that the right initial conditions may be stated e.g. by means of the projector $P_0$ onto $S_0(x_*)$ along $\ker A$.

**Theorem 3.2** *([10], [11]): Let $g \in C^2(\mathcal{D}, \mathbb{R}^m), \mathcal{D} \subset \mathbb{R}^m$ open, $x_* \in \mathcal{D}, g(x_*) = 0, g'(x_*) =: B$. Let the pencil $\lambda A + B$ be regular with index 1, and all its eigenvalues have negative real parts. Then there are a $\tau > 0$, and $\delta(\varepsilon) > 0$ to each $\varepsilon > 0$ such that* *(i) all IVPs for* (3.2) *with*

$$P_0(x(0) - x^0) = 0, \quad |P_0x^0 - P_0x_*| \le \tau$$

*have unique solutions on* $[0, \infty)$,

*(ii)* $|P_0x^0 - P_0x_*| \le \delta(\varepsilon)$ *implies*

$$|x(t; x^0) - x_*| \le \varepsilon, \quad t \ge 0, \quad \textit{and}$$

*(iii)* $|x(t; x^0) - x_*| \longrightarrow 0 \quad (t \to \infty)$.

It should be mentioned that this assertion remains true if $P_0$ is any projector with $\ker P_0 = \ker A$. Comparing with Lyapunov's Theorem above we should take into account that the eigenvalues of the regular zero index pencil $\lambda I + B$ are exactly the eigenvalues of the matrix $-B$.
Clearly, as for regular ODEs, all terms characterizing stability, that is, $A, B, P_0$, are available numerically.
The proof follows the classical lines (e.g. [9]) via linearization at the stationary solution. For the linear part, the decoupling by means of the projector technique sketched in Section 2 has been applied.
Further, let us stress once more that $g \in C^2$ is assumed even in the classical case.

It seems to be straightforward to propose an adequate Caratheodory theory for index 1 DAEs (cf. [13] for first aspects).
Unfortunately, to obtain a similar result also for the index 2 case, we need some structural condition to be sure that the DAE has index 2 around $x_*$ indeed provided the pencil $\lambda A + g'(x_*)$ has.
Let us illustrate this problem by the following example (cf. [12], [6]), which describes a simple nonlinear resistor circuit. The system

$$\text{(3.4)} \qquad \left.\begin{array}{l} x_1' - \alpha(x_3) = 0 \\ x_2' - \beta(x_3) = 0 \\ x_1 + x_2x_3 + x_3^3 = 0 \end{array}\right\}$$

with given smooth functions $\alpha, \beta; \mathbb{R} \to \mathbb{R}$, is easily checked to have index 1 all over $\{x \in \mathbb{R}^3 : x_2 + = 3x_3^2 \neq 0\}$. The surface given by the constraint in (3.4) has a fold. On the fold curve

$$\mathcal{F} := \{x \in \mathbb{R}^3 : x_1 + x_2x_3 + x_3^3 = 0, x_2 + 3x_3^2 = 0\}$$

the pencils $\lambda A + g'(x)$ are regular but they have index 2. Different choices of the functions $\alpha, \beta$ may lead to bifurcations and impasse points, respectively. Hence, that problem (3.4) represents rather an index 1 DAE having singularities on the fold curve. It should not be considered to be an index 2 DAE.
By Example (3.4), the linearization at points on the fold curve is shown to make no sense at all in general. So we need certain additional structure to be sure that the linearization makes sense and the information at the only point $x_*$ will suffice.

**Theorem 3.3** ([6]) *: Let* $g \in C^2(\mathcal{D}, \mathbb{R}^m), \mathcal{D} \subset \mathbb{R}^m$ *open,* $x_* \in \mathcal{D}, g(x_*) = 0, g'(x_*) =: B$*. Let the pencil* $\lambda A + B$ *be regular with index 2, and all its eigenvalues have negative real parts.*
*Additionally, let the condition*

$$\text{(3.5)} \qquad \begin{array}{c} (I - AA^+)\{g(x) - g(P_0x)\} \in im\,(I - AA^+) \\ BQ_0, x \in B(x_*, \sigma), \end{array}$$

*be given for certain* $\sigma > 0$.
*Then there are a* $\tau > 0$*, and* $\delta(\varepsilon) > 0$ *to each* $\varepsilon > 0$ *such that*

*(i) all IVPs for* (3.2) *with*

$$P_0P_1(x(0)-x^0)=0, \quad |P_0P_1x^0-P_0P_1x_*|\le \tau$$

*have unique solutions on* $[0,\infty)$,

*(ii)* $|P_0P_1x^0-P_0P_1x_*|\le \delta(\varepsilon)$ *implies*

$$|x(t;x^0)-x_*|\le \varepsilon, \quad t\ge 0, \quad \textit{and}$$

*(iii)* $|x(t;x^0)-x_*| \longrightarrow 0 \quad (t\to\infty)$.

*Thereby,* $P_0, P_1$ *are projectors,* $\ker A=\ker P_0, \ker(A+BQ_0)=\ker P_1$.

Condition (3.5) means, roughly speaking, that the derivative free part $(I-AA^+)g(x)$ within (3.2) should depend on the nullspace component $Q_0x$ only linearly. In the case of Hessenberg form equations this is given trivially. Therefore, (3.5) covers both linear equations and Hessenberg form ones. Some further generalization is possible but much more technical.
Again, the characteristic terms $A,B,P_0,P_1$ are available, the index of the pencil as well as its regularity may be checked numerically.
Information on the pencil eigenvalues may be obtained directly, but also via the matrix (2.6) by ususal methods.
Finally, turning back to our original equation (1.1) and its enlarged form (1.13),(1.14), we find the following statement that applies to index 1 DAEs whose leading coefficient has a nullspace varying with $x$.

**Corollary 3.4** *(cf.* [10]*): Let* $A\in C^2(\mathcal{D},L(\mathbb{R}^m)), g\in C^2(\mathcal{D},\mathbb{R}^m)$,
$\mathcal{D}\subset\mathbb{R}^m$ *open,* $x_*\in\mathcal{D}, g(x_*)=0$,

$$A(x_*)=:A_0, \quad g'(x_*)=:B,$$

$P_0$ *be a projector,* $\ker P_0=\ker A_0$.
*Additionally, let*

$$im\,A(x)=im\,A(x_*), \quad x\in B(x_*,\sigma). \tag{3.6}$$

*Let the pencil* $\lambda A_0+B$ *be regular with index* 1, *and all its eigenvalues have negative real parts. Then the assertions (i) - (iii) of Theorem* 3.2 *become true if we replace* (3.2) *by* (1.1).

Note that Corollary 3.4 is obtained by applying Theorem 3.3 to the enlarged system (1.13), (1.14) and tracing back the result to (1.1). Thereby, condition (3.6) appears as the reflection of (3.5). It should be stressed further that the projector $P_0$ onto $\ker A_0=\ker A(x_*)$ obviously depends on $x_*$. In case of a constant nullspace $\ker A(x)\equiv N$ we may use a projector independent of the linearization point and drop the structural condition (3.6) again.
Related results on index 3 equations are given in [10], however this applies to constrained multibody systems rather than to circuit simulation.
Considering limit circles makes some more difficulties since now one cannot work locally around a single point, that is, restrict the problem to a single chart. The linearization result given in [14] is hoped to be the right tool to overcome these difficulties.

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